A 3D-PEEC Formulation Based on the Cell Method for Full-Wave Analyses with Conductive, Dielectric, and Magnetic Media

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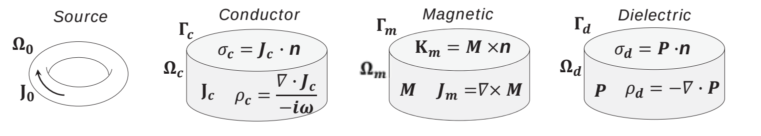
A novel *Partial Element Equivalent Circuit* (PEEC) formulation for solving full-Maxwell's equations, with piecewise homogeneous and linear conductive, dielectric, and magnetic media is presented. It is based on the Cell Method, which by using integral variables as problem unknowns, is naturally suited for developing circuit like approaches such as PEEC. Volume meshing allows complex 3D geometries, with electric and magnetic materials, to be discretized. Electromagnetic couplings in the air domain are modelled by integral equations taking into account the time delay effects on the electromagnetic fields propagation.

Domain Subdivison



$$\Omega = \Omega_c \sqcup \Omega_d \sqcup \Omega_m, \quad \partial\Omega_c = \Gamma_c, \quad \partial\Omega_m = \Gamma_m, \quad \partial\Omega_d = \Gamma_d, \quad \Omega^c = \mathbb{R}^3 \backslash \Omega, \quad \Omega_0 \subset \Omega^c$$

3D-PEEC Formulation

By introducing the electric scalar potential φ, the magnetic vector potential $\boldsymbol{A}$ and the **Lorenz's Gauge** condition, **Maxwell's equations** can be written as:

$$\Delta \boldsymbol{A} + k_0^2 \boldsymbol{A} = -\mu_0(\boldsymbol{J_{eq}} + \boldsymbol{J_0}) \qquad \Delta\varphi + k_0^2\varphi = -\varepsilon_0{}^{-1}\rho_{eq} \tag{1}$$

where $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$ is the wavenumber in homogeneous media while $\boldsymbol{J_{eq}}$ and ρ_{eq} are the **equivalent current** and **charge densities**:

$$\boldsymbol{J_{eq}} = \begin{cases} \boldsymbol{J_c} \\ \boldsymbol{J_p} = i\omega \boldsymbol{P} \\ \boldsymbol{J_m} = \nabla \times \boldsymbol{M} \end{cases} = \boldsymbol{E} \cdot \begin{cases} \rho_c^{-1} & in\ \boldsymbol{\Omega_c} \\ i\omega\varepsilon_0(\varepsilon_r - 1) & in\ \boldsymbol{\Omega_d} \\ i\omega\varepsilon_0(\mu_r - 1) & in\ \boldsymbol{\Omega_m} \end{cases} \tag{2}$$
$$\boldsymbol{J_{eq}} = \boldsymbol{E}\rho_{eq}^{-1}, \qquad \rho_{eq} = -\frac{\nabla \cdot \boldsymbol{J_{eq}}}{i\omega} \quad in\ \boldsymbol{\Omega}$$

The integral solution of equation (1) is:

$$\begin{aligned} \boldsymbol{A}(x) &= \boldsymbol{A}_0(x) + \mu_0 \int_\Omega \boldsymbol{J_{eq}}(y) g(x,y) dy + \mu_0 \int_\Gamma \boldsymbol{K_m}(y) g(x,y) dy \\ \varphi(x) &= \varepsilon_0^{-1} \int_\Omega \rho_{eq}(y) g(x,y) dy + \varepsilon_0^{-1} \int_\Gamma \sigma_{eq}(y) g(x,y) dy \end{aligned} \tag{3}$$

where:

- $g(x,y)$: *retarded Green's function*;
- $\boldsymbol{K_m}$: magnetic surface current density;
- $\sigma_{eq}(= \sigma_c\ in\ \boldsymbol{\Omega_c}, = \sigma_d\ in\ \boldsymbol{\Omega_d})$: equivalent surface charge density;
- $\rho_{eq}(= \rho_c\ in\ \boldsymbol{\Omega_c}, = \rho_d\ in\ \boldsymbol{\Omega_d})$: equivalent volume charge density.

Continuity relationships between current and charge densities:

$$\nabla \cdot \boldsymbol{J_{eq}} + i\omega\rho_{eq} = 0\ in\ \boldsymbol{\Omega}, \quad \nabla \cdot \boldsymbol{K_m} = 0\ \&\ \nabla_\Gamma \cdot \boldsymbol{J_{eq}} + i\omega\sigma_{eq} = 0\ on\ \boldsymbol{\Gamma} \tag{4}$$

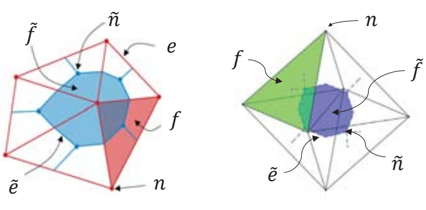
Where $\nabla_\Gamma \cdot$ is the surface divergence operator.

Cell Method Discretization

The tetrahedral discretization of the domain defines a *primal grid* $\boldsymbol{\mathcal{G}_\Omega}$, with n nodes, e edges, f faces and v volumes, and an *augmented dual grid*: $\widetilde{\boldsymbol{\mathcal{G}}}_{\boldsymbol{a}} = \widetilde{\boldsymbol{\mathcal{G}}}_{\boldsymbol{\Omega}} + \widetilde{\boldsymbol{\mathcal{G}}}_{\boldsymbol{\Gamma}}$, $(\tilde{n}, \tilde{e}, \tilde{f}, \tilde{v})$.

The following incidence matrices, describing connectivity between grid cells, are defined as:

- $\boldsymbol{G_\Omega}$, e-n on $\boldsymbol{\mathcal{G}_\Omega}$
- $\boldsymbol{C_\Omega}$, f-e on $\boldsymbol{\mathcal{G}_\Omega}$
- $\boldsymbol{D_\Omega}$, v-f on $\boldsymbol{\mathcal{G}_\Omega}$
- $\tilde{\mathbf{C}}_\Omega = \mathbf{C}_\Omega^{\mathrm{T}}$, $\tilde{f}$-$\tilde{e}$ on $\widetilde{\boldsymbol{\mathcal{G}}}_{\boldsymbol{\Omega}}$
- $\mathbf{C}_{\Gamma_m\Omega_m}$, e on $\boldsymbol{\mathcal{G}_\Gamma}$ - e on $\boldsymbol{\mathcal{G}_\Omega}$
- $\boldsymbol{D}_{\Gamma\Omega}$, f on $\boldsymbol{\mathcal{G}_\Gamma}$ - f on $\boldsymbol{\mathcal{G}_\Omega}$
- $\tilde{\mathbf{G}}_a = -[\mathbf{D}_\Omega^{\mathrm{T}}\ \mathbf{D}_{\Omega\Gamma}^{\mathrm{T}}]$
- $\tilde{\mathbf{C}}_{am} = [\mathbf{C}_{\Omega_m}^{\mathrm{T}}\ \mathbf{C}_{\Gamma_m\Omega_m}^{\mathrm{T}}]$



The following arrays of DoFs are defined on $\boldsymbol{\mathcal{G}_\Omega}$ and $\widetilde{\boldsymbol{\mathcal{G}}}_{\boldsymbol{a}}$:

- $\mathbf{m}$ on $e \in \boldsymbol{\mathcal{G}_{\Omega_m}}$, $m_i = \int_{e_i} \mathbf{M} \cdot d\boldsymbol{l}$
- $\boldsymbol{j_{eq}}$ on $f \in \boldsymbol{\mathcal{G}_\Omega}$, $j_{eq_i} = \int_{f_i} \boldsymbol{J_{eq}} \cdot d\boldsymbol{S}$
- $\boldsymbol{q_{eq}}$ on $v \in \boldsymbol{\mathcal{G}_\Omega}$, $q_i = \int_{v_i} \rho_{eq}\, dV$
- $\boldsymbol{k_m}$ on $e \in \boldsymbol{\mathcal{G}_{\Gamma_m}}$, $k_{m_i} = \int_{e_i} \boldsymbol{K_m} \cdot d\boldsymbol{l}$
- $\boldsymbol{\sigma_{eq}}$ on $f \in \boldsymbol{\mathcal{G}_\Gamma}$, $\sigma_{eq_i} = \int_{f_i} \sigma_{eq}\, dS$

- $\widetilde{\boldsymbol{\Phi}}$ on $\tilde{n} \in \widetilde{\boldsymbol{\mathcal{G}}}_{\boldsymbol{a}}$, $\widetilde{\Phi}_i = \varphi(x_{\tilde{n}_i})$
- $\tilde{\mathbf{a}}_{\boldsymbol{a}}$ on $\tilde{e} \in \widetilde{\boldsymbol{\mathcal{G}}}_{\boldsymbol{a}}$, $\tilde{a}_{a_i} = \int_{\tilde{e}_i} \boldsymbol{A} \cdot d\boldsymbol{l}$
- $\tilde{\mathbf{e}}$ on $\tilde{e} \in \widetilde{\boldsymbol{\mathcal{G}}}_{\boldsymbol{\Omega}}$, $\tilde{e}_i = \int_{\tilde{e}_i} \boldsymbol{E} \cdot d\boldsymbol{l}$
- $\tilde{\boldsymbol{b}}$ on $\tilde{f} \in \widetilde{\boldsymbol{\mathcal{G}}}_{\boldsymbol{\Omega}}$, $\tilde{b}_i = \int_{f_i} \boldsymbol{B} \cdot d\boldsymbol{S}$

Algebraic System

The electric and magnetic constitutive relationships in weak form are:

$$\int_\Omega \mathrm{w}_i^f(x) \cdot \left(\mathbf{E}(x) - \varrho_{eq}\, \mathbf{J_{eq}}(x)\right) dx = 0 \qquad \int_{\Omega_m} \mathrm{w}_i^e(x) \cdot \left(\mathbf{B}(x) - \hat{\mu}\, \mathbf{M}(x)\right) dx = 0 \tag{5}$$

Where w_i^f and w_i^e are *face and edge shape functions* while $\hat{\mu} = (\mu_0\mu_r)/(\mu_r - 1)$.

By expanding $\mathbf{J_{eq}}$ and $\mathbf{M}$ with w_i^f and w_i^e, and by writing $\tilde{\mathbf{e}} = -i\omega\tilde{\mathbf{a}} - \tilde{\mathbf{G}}_a\widetilde{\boldsymbol{\Phi}}_a$ and $\tilde{\mathbf{b}} = \tilde{\mathbf{C}}_{am}\tilde{\mathbf{a}}_a$, the following system can be obtained:

$$\begin{bmatrix} \mathbf{R} + i\omega\mathbf{L}_{\Omega,\Omega} & i\omega\mathbf{L}_{\Omega,\Gamma_m}\mathbf{C}_{\Gamma_m,\Omega_m} & \tilde{\mathbf{G}}_a \\ \tilde{\mathbf{C}}_{am}\mathbf{L}_{\Omega\cup\Gamma_m,\Omega} & \mathbf{S} - \tilde{\mathbf{C}}_{am}\mathbf{L}_{\Omega\cup\Gamma_m,\Gamma_m}\mathbf{C}_{\Gamma_m,\Omega_m} & \mathbb{O} \\ \tilde{\mathbf{G}}_a^{\mathrm{T}} & \mathbb{O} & -i\omega\mathbf{P}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{j_{eq}} \\ \mathbf{m} \\ \widetilde{\boldsymbol{\Phi}}_a \end{bmatrix} = \begin{bmatrix} -i\omega\tilde{\mathbf{a}}_0 \\ \tilde{\mathbf{C}}_{am}\tilde{\mathbf{a}}_0 \\ \mathbb{O} \end{bmatrix} \tag{6}$$

Where $\mathbf{R}$ and $\mathbf{S}$ are the constitutive matrices while $\mathbf{L}$ and $\mathbf{P}$ are full matrices called "*inductance*" matrix and "*partial coefficient of potential*" matrix:

$$\begin{bmatrix} \tilde{\mathbf{a}}_\Omega \\ \tilde{\mathbf{a}}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{\Omega,\Omega} & \mathbf{L}_{\Omega,\Gamma} \\ \mathbf{L}_{\Gamma,\Omega} & \mathbf{L}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} \boldsymbol{j_{eq}} \\ \boldsymbol{k_m} \end{bmatrix} \qquad \begin{bmatrix} \widetilde{\boldsymbol{\Phi}}_\Omega \\ \widetilde{\boldsymbol{\Phi}}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{\Omega,\Omega} & \mathbf{P}_{\Omega,\Gamma} \\ \mathbf{P}_{\Gamma,\Omega} & \mathbf{P}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} \boldsymbol{q_{eq}} \\ \boldsymbol{\sigma_{eq}} \end{bmatrix} \tag{7}$$

Finally, by applying $\mathbf{Q}$ as defined in (8), the initial system $\mathbf{A\,x} = \mathbf{b}$ (6) with $f + e|_{\mathcal{G}_{\Omega_m}} + \tilde{n}$ DoFs, can be projected into a new system of equations: $(\mathbf{Q}^{\mathrm{T}}\mathbf{AQ})\,\mathbf{x} = \mathbf{Q}^{\mathrm{T}}\mathbf{b}$. This operation **reduces the DoFs** to $f|_{\mathcal{G}_{\Omega_c}} + e|_{\mathcal{G}_{\Omega_d}} + e|_{\mathcal{G}_{\Omega_m}} + n|_{\mathcal{G}_{\Omega_m}} + \tilde{n}$. Instead of (6) the new system severely enforces equations (4) and makes the formulation consistent for the **whole range of frequency**.

$$\begin{bmatrix} \mathbf{j}_c \\ \mathbf{j}_d \\ \mathbf{j}_m \\ \mathbf{m} \\ \widetilde{\boldsymbol{\Phi}}_a \end{bmatrix} = \begin{bmatrix} \mathbb{I} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & i\omega\,\mathbf{C}_{\Omega_d} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & i\omega\,\mathbf{C}_{\Omega_d} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & i\omega\,\mathbb{I} & \mathbf{G}_{\Omega_m} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{I} \end{bmatrix} \begin{bmatrix} \mathbf{j}_c \\ \mathbf{u}_d \\ \mathbf{u}_m \\ \boldsymbol{\psi} \\ \widetilde{\boldsymbol{\Phi}}_a \end{bmatrix} \tag{8}$$

Numerical Results

A **3D-PEEC** code has been developed using FORTRAN® for implementing matrix routines and MATLAB® for the final system assembly.
Both volume and surface elements have been implemented for the discretization of the conductive domain in order to strongly reduce the number of DoFs at high frequency, when the skin effect is considerable.
A «UHF wireless power transfer» has been considered for the validation: two loop antennas printed on a dielectric substrate ($\varepsilon_r = 2.1$) with a magnetic substrate ($\mu_r = 1000$) as flux concentrator, *Fig.1*.

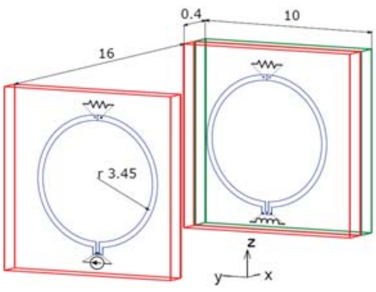


Fig.1: dimensions in cm.

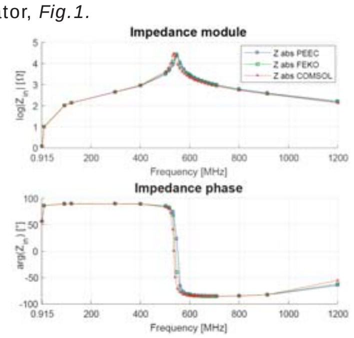


Fig.2.

The results in terms of equivalent input impedance for a frequency range from $915Hz$ to $1.2GHz$ have been compared with the ones given by two commercial software: COMSOL® and FEKO®, *Fig.2*.

Table 1	3D-PEEC	COMSOL®	FEKO®
Tetrahedra	9,919	239,842	12,922
Triangles	1,968	–	3,830
DoFs (assembled)	31,792	4,773,558	35,637
DoFs (solved)	15,009	4,773,558	35,637
Time (assembling) [s]	760.93	–	776.54
Time (solving) [s]	184.03	–	2,150.49
Time (total) [s]	944.96	2,148	2,927.03

The number of unknowns, the computational time and the details of the models are reported in *Table 1*. This work is supported by the BIRD162948/1 grant of the Department of industrial Engineering, University of Padova.